<u>Worksheet: Newton's Mentors</u>

Unit: Infinite Sequences and Series (AP Calc BC Unit 10)

Topics: Taylor & Maclaurin Series, Interval of Convergence, Error Bounds

In this worksheet, students explore the mathematical legacy of Newton's mentors, Barrow, Wallis, and Descartes, by constructing Maclaurin series to approximate integrals, using the binomial series to derive a representation of π, and building Taylor polynomials with error bounds to approximate trigonometric functions.

Part 1: Barrow – Area, Integrals, and Series

Historically, Barrow focused on finding the area under a curve, laying the groundwork for the Fundamental Theorem of Calculus.

Consider the function $f(x) = \frac{sin(x^2)}{x}$.

1. Explain why $f(x) = \frac{sin(x^2)}{x}$ is not defined at x = 0, and determine $\lim_{x \to 0} f(x)$ and define $f(0)$ so that f is continuous.

2. Write the first four non-zero terms and the general term of the Maclaurin series for y = sin(x).

3. Use your answer from Question 2 to find the first four non-zero terms and the general term of the Maclaurin series for $f(x) = \frac{sin(x^2)}{x}$.

4. Determine the interval of convergence for the series found in Question 3. Show the work that leads to your conclusion.

5. Using the first three non-zero terms of your series, approximate the definite integral $\int_0^1 \frac{sin(x^2)}{x} dx$.

6. Use the Alternating Series Error Bound to determine the maximum possible error of the approximation you found in Question 5.

Part 2: Wallis – The Binomial Series and Approximation

Wallis is famous for his Wallis product, an infinite product formula for $\frac{\pi}{2}$, and his pioneering work extending algebraic patterns to fractional exponents. His insights inspired Newton to develop the general binomial series expansion for non-integer powers, which became a powerful tool for computing digits of π.

Consider the function $g(x) = \sqrt{1 - x^2}$.

1. Use the general binomial series expansion to write the first four non-zero terms and the general term of the Maclaurin series for $g(x) = (1 - x^2)^{1/2}$

2. Using the first four non-zero terms of your series, approximate the definite integral $\int_0^{0.5} \sqrt{1 - x^2} dx$

3. Geometrically, $\int_0^1 \sqrt{1 - x^2}\,dx$ represents the area of a quarter circle of radius 1, which equals $\frac{\pi}{4}$. Explain why integrating the binomial series for $(1 - x^2)^{1/2}$ term-by-term from 0 to 1 produces a series representation for $\frac{\pi}{4}$. Write out the first four terms of this series for $\frac{\pi}{4}$.

Part 3: Descartes – Analytic Geometry and Taylor Polynomials

Much of Descartes's work was on uniting algebra and geometry, making it possible to describe curves with equations and use algebraic methods to study their properties. Taylor polynomials extend this idea by approximating transcendental curves using polynomial equations.

Consider the curve $y = cos(x)$, centered at x = $\frac{\pi}{3}$.

1. Find the 1st degree Taylor polynomial, $P_1(x)$, and the 2nd degree Taylor polynomial, $P_2(x)$, for y = cos(x) centered at x = $\frac{\pi}{3}$.

2. Explain how $P_1(x)$ represents a tangent line approximation to the curve, and how $P_2(x)$ improves this by acting as an "approximating parabola."

3. Use $P_2(x)$ to approximate cos(1).

4. Use the Lagrange Error Bound to show that your approximation for cos(1) is within 0.001 of the exact value.