

Worksheet: Cauchy's Calculus

Unit: Limits and Continuity (AP Calc AB Unit 1)

Topics: Limits, Convergence, Continuity, Intermediate Value Theorem & Squeeze Theorem

In this worksheet, students trace the historical development of limits and continuity through Cauchy's original definitions, computing sequences to explore convergence, classifying discontinuities in piecewise functions, and applying the Intermediate Value Theorem and Squeeze Theorem to guarantee the existence of roots and evaluate limits.

Part I: Limits

Before Cauchy, mathematicians used limits loosely, speaking of quantities approaching a value without specifying precisely what that meant. Cauchy changed this by giving limits a clear written definition in his *Cours d'analyse*, which later mathematicians formalized into the rigorous language we use today.



1. Cauchy defined limits as follows:

“When the values successively attributed to a particular variable indefinitely approach a fixed value in such a way as to end up by differing from it by as little as we wish, this fixed value is called the limit of all the other values.”

a) In your own words, explain what Cauchy means. What does it mean for a variable to ‘indefinitely approach’ a fixed value? What does ‘differing from it by as little as we wish’ tell us about how close that variable gets to that value?

b) Let $a_n = \frac{2n+1}{n+1}$. Compute a_1 , a_{10} , a_{100} , and a_{1000} . Based on the pattern, what value does the sequence appear to approach?

c) Confirm your answer to (b) algebraically by evaluating $\lim_{n \rightarrow \infty} \frac{2n+1}{n+1}$. Show your work

- d) How close is a_{100} to the limit you found? How close is a_{1000} ? Explain how this illustrates Cauchy's phrase 'differing from it by as little as we wish'.

2. Cauchy also described variables that shrink to zero: what he called infinitesimals:

When the successive absolute values of a variable decrease indefinitely in such a way as to become less than any given quantity, that variable becomes what is called an infinitesimal. Such a variable has zero for its limit.

- a) In modern language, what is Cauchy describing? How does this relate to saying that a sequence has a limit of 0?
- b) For each sequence below, compute the first four terms for each. Then, determine whether it is an infinitesimal in Cauchy's sense (whether its limit is 0).

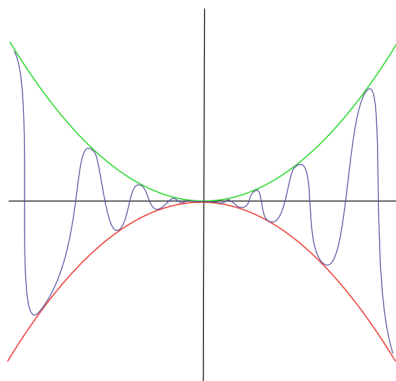
(i) $a_n = \frac{1}{n^2}$

(ii) $b_n = \frac{(-1)^n}{n}$

(iii) $c_n = (-1)^n$

- c) Sequence ii alternates in sign but still qualifies as an infinitesimal. Explain why the alternating sign does not prevent b_n from being an infinitesimal, while c_n fails the definition.

3. Cauchy frequently used bounding arguments: trapping a quantity between two values that both approach the same limit.



- a) Suppose $-x^2 \leq f(x) \leq x^2$ for all x near 0. What can you conclude about $\lim_{x \rightarrow 0} f(x)$? Explain why and identify the general theorem that justifies your conclusion.

- b) Use this theorem to evaluate $\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right)$. Clearly identify the bounding functions and show that the conditions of the theorem are met.

Part II: Continuity

Before Cauchy, mathematicians treated continuity intuitively. A function was ‘continuous’ if you could draw it without lifting your pen. Cauchy replaced this with a precise condition relating small changes in input to small changes in output.

4. Cauchy defined continuity as follows:

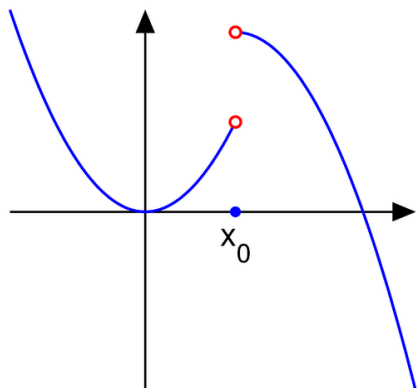
“The function $f(x)$ is continuous with respect to x between the given limits if, between these limits, an infinitely small increment in the variable always produces an infinitely small increment in the function itself”

- a) In your own words, explain what Cauchy means. What is the ‘infinitely small increment in the variable’ and what does it mean for the function to respond with an ‘infinitely small increment’ of its own?
- b) In modern terms, a function f is continuous at $x = a$ if three conditions are met:
- i) $f(a)$ is defined
 - ii) $\lim_{x \rightarrow a} f(x)$ exists
 - iii) $\lim_{x \rightarrow a} f(x) = f(a)$.

Explain how each of these three conditions connects to Cauchy’s definition above

- c) Verify that $f(x) = x^2 + 3x$ is continuous at $x = 1$ by checking all three conditions.

5. Cauchy's definition also tells us what it means for a function to fail to be continuous: a small change in input produces a sudden, non-small change in output.



a) Define a jump discontinuity using one-sided limits

b) Consider the piecewise function defined by

$$f(x) = x + 2 \text{ when } x < 2 \text{ and } f(x) = x^2 - 1 \text{ when } x \geq 2.$$

Compute $\lim_{x \rightarrow 2^-} f(x)$ and $\lim_{x \rightarrow 2^+} f(x)$. Is f continuous at $x = 2$?

If not, classify the discontinuity

c) Explain in Cauchy's terms why this function fails to be continuous at $x = 2$. What happens to the 'increment in the function' as the input crosses 2?

6. Cauchy also studied functions whose behavior becomes unbounded near certain points.

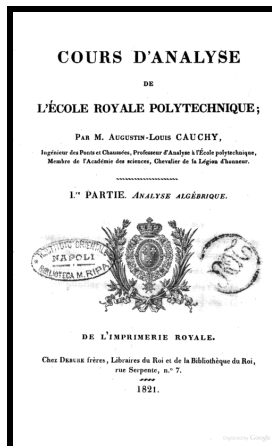
Consider $f(x) = \frac{1}{x}$

a) Is f continuous at $x = 0.01$? Check the three conditions for continuity

b) What happens to $f(x)$ as $x \rightarrow 0$ from the right? From the left? What type of discontinuity occurs at $x = 0$?

7. In the *Cours d'analyse*, Cauchy argued that a continuous function that is positive at one point and negative at another must equal zero somewhere in between.

a) Suppose f is continuous on $[0, 5]$, with $f(0) = -2$ and $f(5) = 8$. Can you guarantee that $f(x) = 3$ for some x in $[0, 5]$? What about $f(x) = 10$? Explain your reasoning, and state the theorem you are using.



b) Let $f(x) = x^3 - 4x + 1$. Show that f has a root in the interval $[1,2]$. Identify which conditions of the theorem you identified above you are checking, and explain why each is satisfied.