Solutions: Newton's Mentors

Part 1: Barrow – Area, Integrals, and Series

Consider

$$f(x) = \frac{\sin(x^2)}{x}.$$

1. Continuity at $x = 0$

The function

$$f(x) = \frac{\sin(x^2)}{x}$$

is not defined at $x = 0$ because the denominator is 0.

To make f continuous at $x = 0$, compute

$$\lim_{x \to 0} \frac{\sin(x^2)}{x}.$$

Since $\sin(x^2) \sim x^2$ as $x \to 0$,

$$\frac{\sin(x^2)}{x} \sim \frac{x^2}{x} = x \to 0.$$

Therefore,

$$\lim_{x \to 0} \frac{\sin(x^2)}{x} = 0.$$

So we define

$$\boxed{f(0) = 0}$$

to make f continuous.

2. Maclaurin series for $\sin x$

The Maclaurin series for $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

So the first four nonzero terms are

$$\boxed{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}}$$

and the general term is

$$\boxed{\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}.$$

3. Maclaurin series for $\dfrac{\sin(x^2)}{x}$

Substitute x^2 into the series for $\sin x$:

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$$

Now divide by x:

$$\frac{\sin(x^2)}{x} = x - \frac{x^5}{3!} + \frac{x^9}{5!} - \frac{x^{13}}{7!} + \cdots$$

So the first four nonzero terms are

$$\boxed{x - \frac{x^5}{6} + \frac{x^9}{120} - \frac{x^{13}}{5040}}$$

and the general term is

$$\boxed{\frac{\sin(x^2)}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(2n+1)!}}.$$

4. Interval of convergence

Consider the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(2n+1)!}.$$

Use the Ratio Test:

$$\left| \frac{x^{4(n+1)+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{4n+1}} \right| = \left| \frac{x^4}{(2n+3)(2n+2)} \right|.$$

As $n \to \infty$,

$$\left| \frac{x^4}{(2n+3)(2n+2)} \right| \to 0$$

for every real x. Therefore the series converges for all real x.

So the interval of convergence is

$$\boxed{(-\infty, \infty)}.$$

5. Approximate $\displaystyle\int_0^1 \frac{\sin(x^2)}{x}\,dx$

Using the first three nonzero terms,

$$\frac{\sin(x^2)}{x} \approx x - \frac{x^5}{6} + \frac{x^9}{120}.$$

Then

$$\int_0^1 \frac{\sin(x^2)}{x}\,dx \approx \int_0^1 \left(x - \frac{x^5}{6} + \frac{x^9}{120}\right)\,dx.$$

Compute:

$$\int_0^1 x\,dx = \frac{1}{2}, \qquad \int_0^1 \frac{x^5}{6}\,dx = \frac{1}{36}, \qquad \int_0^1 \frac{x^9}{120}\,dx = \frac{1}{1200}.$$

So

$$\int_0^1 \frac{\sin(x^2)}{x}\,dx \approx \frac{1}{2} - \frac{1}{36} + \frac{1}{1200}.$$

Thus

$$\boxed{\int_0^1 \frac{\sin(x^2)}{x}\,dx \approx 0.473056}$$

(to six decimal places).

6. Alternating Series Error Bound

The next omitted term in the integrand is

$$-\frac{x^{13}}{7!}.$$

So the error in the integral is at most

$$\int_0^1 \frac{x^{13}}{7!}\,dx = \frac{1}{7!}\cdot\frac{1}{14} = \frac{1}{70560}.$$

Therefore the maximum possible error is

$$\boxed{\frac{1}{70560} \approx 1.42 \times 10^{-5}}.$$

Part 2: Wallis – The Binomial Series and Approximation

Consider

$$g(x) = \sqrt{1-x^2} = (1-x^2)^{1/2}.$$

1. Maclaurin series for $\sqrt{1-x^2}$

Using the binomial series

$$(1+u)^k = \sum_{n=0}^{\infty} \binom{k}{n} u^n, \qquad |u| < 1,$$

with $u = -x^2$ and $k = \frac{1}{2}$,

$$(1-x^2)^{1/2} = 1 + \sum_{n=1}^{\infty} \binom{1/2}{n} (-x^2)^n.$$

The first four nonzero terms are

$$\boxed{1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16}}$$

(and the next term is $-\frac{5x^8}{128}$).

A convenient general form is

$$\boxed{\sqrt{1-x^2} = 1 - \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n(2n-1)} x^{2n}}$$

for $|x| < 1$.

2. Approximate $\int_0^{0.5} \sqrt{1-x^2}\, dx$

Using the first four nonzero terms,

$$\sqrt{1-x^2} \approx 1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16}.$$

Then

$$\int_0^{0.5} \sqrt{1-x^2}\, dx \approx \int_0^{0.5} \left(1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16}\right) dx.$$

Integrate term-by-term:

$$\int \left(1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16}\right) dx = x - \frac{x^3}{6} - \frac{x^5}{40} - \frac{x^7}{112}.$$

Evaluate from 0 to 0.5:

$$0.5 - \frac{(0.5)^3}{6} - \frac{(0.5)^5}{40} - \frac{(0.5)^7}{112}.$$

So

$$\boxed{\int_0^{0.5} \sqrt{1-x^2}\, dx \approx 0.478316}$$

(to six decimal places).

3. Series for $\frac{\pi}{4}$

Geometrically,

$$\int_0^1 \sqrt{1-x^2}\,dx$$

is the area of a quarter circle of radius 1, so

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}.$$

Since

$$\sqrt{1-x^2} = 1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \frac{5x^8}{128} - \cdots,$$

integrating term-by-term from 0 to 1 gives

$$\frac{\pi}{4} = \int_0^1 \left(1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \cdots\right) dx.$$

Therefore,

$$\frac{\pi}{4} = 1 - \frac{1}{6} - \frac{1}{40} - \frac{1}{112} - \frac{5}{1152} - \cdots$$

So the first four terms of the series for $\frac{\pi}{4}$ are

$$\boxed{\frac{\pi}{4} \approx 1 - \frac{1}{6} - \frac{1}{40} - \frac{1}{112} - \cdots}$$

Part 3: Descartes – Analytic Geometry and Taylor Polynomials

Consider $f(x) = \cos x$, centered at

$$a = \frac{\pi}{3}.$$

1. Find $P_1(x)$ and $P_2(x)$

We need:

$$f(x) = \cos x, \qquad f'(x) = -\sin x, \qquad f''(x) = -\cos x.$$

At $x = \frac{\pi}{3}$,

$$f\left(\frac{\pi}{3}\right) = \frac{1}{2}, \qquad f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}, \qquad f''\left(\frac{\pi}{3}\right) = -\frac{1}{2}.$$

So the first-degree Taylor polynomial is

$$P_1(x) = f(a) + f'(a)(x-a) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right).$$

Thus

$$\boxed{P_1(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)}$$

The second-degree Taylor polynomial is

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2.$$

So

$$P_2(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2.$$

Thus

$$\boxed{P_2(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2}$$

2. Interpretation

The polynomial $P_1(x)$ is the tangent line approximation to $\cos x$ at $x = \frac{\pi}{3}$, because it uses only the function value and slope there.

The polynomial $P_2(x)$ improves on this by also incorporating the second derivative, so it captures the curve's concavity. That is why it acts like an "approximating parabola" and usually gives a better local approximation.

3. Approximate $\cos(1)$ using $P_2(x)$

Substitute $x = 1$ into $P_2(x)$:

$$P_2(1) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(1 - \frac{\pi}{3}\right) - \frac{1}{4}\left(1 - \frac{\pi}{3}\right)^2.$$

Numerically,

$$\boxed{P_2(1) \approx 0.540317}$$

So

$$\boxed{\cos(1) \approx 0.540317}$$

using the quadratic Taylor approximation.

4. Lagrange Error Bound

For a second-degree Taylor polynomial, the remainder satisfies

$$|R_2(x)| \le \frac{M}{3!}|x-a|^3,$$

where M is an upper bound for $|f^{(3)}(x)|$ on the interval between $x = 1$ and $x = \frac{\pi}{3}$.

Since

$$f^{(3)}(x) = \sin x,$$

we may take $M = 1$, because $|\sin x| \leq 1$.

Thus

$$|R_2(1)| \leq \frac{1}{6}\left|1 - \frac{\pi}{3}\right|^3.$$

Now

$$\left|1 - \frac{\pi}{3}\right| \approx 0.0472,$$

so

$$|R_2(1)| \leq \frac{(0.0472)^3}{6} \approx 1.75 \times 10^{-5}.$$

Therefore,

$$\boxed{|R_2(1)| < 0.001.}$$

So the approximation from Question 3 is definitely within 0.001 of the exact value of cos(1).