Solutions: High School Math 100 Years Ago

Logarithms

1. **Find the logarithm of 34,237.**

   The required mantissa is (§421) the same as the mantissa for 3423.7; therefore, it will be found by adding to the mantissa for 3423 seven tenths of the difference between the mantissas for 3423 and 3424.

   The mantissa for 3423 is 53441.

   The difference between the mantissas for 3423 and 3424 is 12.

   Hence, the mantissa for 3423.7 is $53441 + (0.7 \text{ of } 12) = 53449$.

   Therefore, the logarithm of 34,237 is 4.53449.

2. **Find the logarithm of 0.0015764.**

   The required mantissa is the same as the mantissa for 1576.4.

   The mantissa for 1576 is 19756.

   The difference between the mantissas for 1576 and 1577 is 27.

   Hence, the mantissa for 1576.4 is $19756 + (0.4 \text{ of } 27) = 19767$.

   Therefore, the logarithm of 0.0015764 is $7.19767 - 10$.

3. **Find the logarithm of 32.6708.**

   The required mantissa is the same as the mantissa for 3267.08.

   The mantissa for 3267 is 51415.

   The difference between the mantissas for 3267 and 3268 is 13.

   Hence, the mantissa for 3267.08 is $51415 + (0.08 \text{ of } 13) = 51416$.

   Therefore, the logarithm of 32.6708 is 1.51416.

Ratios and Proportions

Given

$$a : b = c : d,$$

we have

$$\frac{a}{b} = \frac{c}{d}.$$

1. **Show that $ac : bd = c^2 : d^2$.**

   Since

   $$\frac{a}{b} = \frac{c}{d},$$

multiplying both sides by $\frac{c}{d}$ gives

$$\frac{ac}{bd} = \frac{c^2}{d^2}.$$

Hence

$$ac : bd = c^2 : d^2.$$

2. **Show that $ab : cd = a^2 : c^2$.**

Since

$$\frac{a}{b} = \frac{c}{d},$$

multiplying both sides by $\frac{b}{c}$ gives

$$\frac{ab}{cd} = \frac{a^2}{c^2}.$$

Hence

$$ab : cd = a^2 : c^2.$$

3. **Show that $a^2 - b^2 : c^2 - d^2 = a^2 : c^2$.**

From

$$\frac{a}{b} = \frac{c}{d},$$

let

$$\frac{a}{c} = \frac{b}{d} = k.$$

Then

$$a = kc, \qquad b = kd.$$

So

$$a^2 - b^2 = k^2(c^2 - d^2).$$

Therefore

$$\frac{a^2 - b^2}{c^2 - d^2} = k^2 = \frac{a^2}{c^2},$$

and hence

$$a^2 - b^2 : c^2 - d^2 = a^2 : c^2.$$

4. **Show that $2a + b : 2c + d = b : d$.**

Again let

$$a = kc, \qquad b = kd.$$

Then

$$2a + b = 2kc + kd = k(2c + d).$$

Hence

$$\frac{2a + b}{2c + d} = k = \frac{b}{d}.$$

Therefore

$$2a + b : 2c + d = b : d.$$

5. **Show that** $5a - b : 5c - d = a : c$**.**

   Using
   $$a = kc, \qquad b = kd,$$
   we get
   $$5a - b = 5kc - kd = k(5c - d).$$
   Thus
   $$\frac{5a - b}{5c - d} = k = \frac{a}{c}.$$
   Therefore
   $$5a - b : 5c - d = a : c.$$

The Binomial Theorem

Recall that
$$(1 + t)^n = 1 + nt + \frac{n(n-1)}{2!}t^2 + \frac{n(n-1)(n-2)}{3!}t^3 + \cdots$$

1. **Expand** $(1 + x)^{-1}$ **to four terms.**

   Here $n = -1$ and $t = x$, so
   $$(1 + x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$
   Thus the first four terms are
   $$\boxed{1 - x + x^2 - x^3.}$$

2. **Expand** $(1 - x)^{1/2}$ **to four terms.**

   Here $n = \frac{1}{2}$ and $t = -x$, so
   $$(1 - x)^{1/2} = 1 + \frac{1}{2}(-x) + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}(-x)^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}(-x)^3 + \cdots$$
   $$= 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} + \cdots$$
   Hence
   $$\boxed{(1 - x)^{1/2} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} + \cdots}$$

3. **Find the fourth term of** $\left(a - \frac{3}{2\sqrt{x}}\right)^{1/2}$**.**

   Write this as
   $$a^{1/2}\left(1 - \frac{3}{2a\sqrt{x}}\right)^{1/2}.$$

The fourth term comes from the t^3 term:

$$a^{1/2} \cdot \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(-\frac{3}{2a\sqrt{x}}\right)^3.$$

Now

$$\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!} = \frac{1}{16},$$

so the fourth term is

$$a^{1/2} \cdot \frac{1}{16}\left(-\frac{27}{8a^3x^{3/2}}\right) = -\frac{27}{128a^{5/2}x^{3/2}}.$$

Therefore the fourth term is

$$\boxed{-\frac{27}{128a^{5/2}x^{3/2}}.}$$

4. **Find the fifth term of** $\frac{1}{\sqrt[3]{(a-2x)^2}}$**.**

We have

$$\frac{1}{\sqrt[3]{(a-2x)^2}} = (a-2x)^{-2/3} = a^{-2/3}\left(1-\frac{2x}{a}\right)^{-2/3}.$$

The fifth term comes from the t^4 term:

$$a^{-2/3} \cdot \frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)\left(-\frac{11}{3}\right)}{4!}\left(-\frac{2x}{a}\right)^4.$$

Compute the coefficient:

$$\frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)\left(-\frac{11}{3}\right)}{4!} = \frac{110}{243}.$$

So the fifth term is

$$a^{-2/3} \cdot \frac{110}{243} \cdot \frac{16x^4}{a^4} = \frac{1760x^4}{243a^{14/3}}.$$

Therefore the fifth term is

$$\boxed{\frac{1760x^4}{243a^{14/3}}.}$$

5. **Find the third term of** $(4-7x)^{2/7}$**.**

Write

$$(4-7x)^{2/7} = 4^{2/7}\left(1-\frac{7x}{4}\right)^{2/7}.$$

The third term comes from the t^2 term:

$$4^{2/7} \cdot \frac{\frac{2}{7}\left(-\frac{5}{7}\right)}{2!}\left(-\frac{7x}{4}\right)^2.$$

Since

$$\frac{\frac{2}{7}\left(-\frac{5}{7}\right)}{2!} = -\frac{5}{49},$$

this becomes

$$4^{2/7}\left(-\frac{5}{49}\right)\left(\frac{49x^2}{16}\right) = -\frac{5}{16}4^{2/7}x^2.$$

Also,

$$\frac{4^{2/7}}{16} = 4^{2/7-2} = 4^{-12/7}.$$

So the third term is

$$\boxed{-5 \cdot 4^{-12/7}x^2.}$$

Determinants

1.

$$\begin{vmatrix} 3 & 2 & -2 & 3 \\ 4 & 3 & 7 & -2 \\ 5 & 1 & 2 & 3 \\ 6 & 2 & -3 & 1 \end{vmatrix} = \boxed{-397}$$

2.

$$\begin{vmatrix} 4 & 2 & -1 & 17 \\ 2 & 9 & 0 & 6 \\ 1 & 1 & 0 & 2 \\ 2 & 3 & 0 & 4 \end{vmatrix} = \boxed{-2}$$

Geometry

1. **If a quadrilateral has each side tangent to a circle, prove that the sum of one pair of opposite sides equals the sum of the other pair.**

   Let the quadrilateral be $ABCD$, and suppose the circle touches AB, BC, CD, DA at P, Q, R, S respectively.

   Tangents drawn from the same external point are equal, so

   $$AP = AS, \qquad BP = BQ, \qquad CQ = CR, \qquad DR = DS.$$

   Then

   $$AB = AP + PB, \qquad BC = BQ + QC, \qquad CD = CR + RD, \qquad DA = DS + SA.$$

   Therefore

   $$AB + CD = (AP + PB) + (CR + RD).$$

   Using the equal tangent segments,

   $$AB + CD = (AS + BQ) + (CQ + DS) = BC + DA.$$

Hence

$$\boxed{AB + CD = BC + DA.}$$

2. **The hexagon here has each side tangent to the circle. Prove that $AB+CD+EF = BC + DE + FA$.**

   Let the circle touch AB, BC, CD, DE, EF, FA at points so that the two tangent segments from each vertex are equal. Let these equal tangent lengths from A, B, C, D, E, F be

   $$x_1, x_2, x_3, x_4, x_5, x_6$$

   respectively.

   Then

   $$AB = x_1 + x_2, \quad BC = x_2 + x_3, \quad CD = x_3 + x_4,$$
   $$DE = x_4 + x_5, \quad EF = x_5 + x_6, \quad FA = x_6 + x_1.$$

   So

   $$AB + CD + EF = (x_1 + x_2) + (x_3 + x_4) + (x_5 + x_6),$$

   while

   $$BC + DE + FA = (x_2 + x_3) + (x_4 + x_5) + (x_6 + x_1).$$

   Both sums are

   $$x_1 + x_2 + x_3 + x_4 + x_5 + x_6.$$

   Therefore

   $$\boxed{AB + CD + EF = BC + DE + FA.}$$

3. **Prove that if a quadrilateral has each side tangent to a circle and if the vertices are joined to the center, the sum of the angles at the center opposite any two sides is equal to a straight angle.**

   Let $ABCD$ be the quadrilateral and O the center of the inscribed circle. Since the circle is tangent to all four sides, O lies on the angle bisector of each angle of the quadrilateral. Hence

   $$\angle OAB = \frac{A}{2}, \qquad \angle OBA = \frac{B}{2}.$$

   In triangle AOB,

   $$\angle AOB = 180^\circ - \frac{A+B}{2}.$$

   Similarly, in triangle COD,

   $$\angle COD = 180^\circ - \frac{C+D}{2}.$$

   Adding,

   $$\angle AOB + \angle COD = 360^\circ - \frac{A+B+C+D}{2}.$$

But the sum of the interior angles of a quadrilateral is

$$A + B + C + D = 360°.$$

Therefore

$$\angle AOB + \angle COD = 360° - \frac{360°}{2} = 180°.$$

Thus the sum of the two angles at the center opposite one pair of opposite sides is a straight angle. By the same argument,

$$\angle BOC + \angle DOA = 180°.$$