

# Solutions: Finding Neptune

## Part I: Modeling and Comparing Orbits

Uranus is modeled parametrically by

$$x(t) = 19.2 \cos t - 0.9, \quad y(t) = 19.18 \sin t,$$

and in polar form by

$$r(\theta) = \frac{a(1 - e^2)}{1 + e \cos \theta}, \quad a = 19.2, \quad e = 0.047.$$

### A. Velocity along the orbit

Differentiate:

$$\frac{dx}{dt} = -19.2 \sin t, \quad \frac{dy}{dt} = 19.18 \cos t.$$

So the velocity vector is

$$\vec{v}(t) = \langle -19.2 \sin t, 19.18 \cos t \rangle.$$

At  $t = \frac{\pi}{4}$ ,

$$\vec{v}\left(\frac{\pi}{4}\right) = \left\langle -19.2 \cdot \frac{\sqrt{2}}{2}, 19.18 \cdot \frac{\sqrt{2}}{2} \right\rangle = \langle -9.6\sqrt{2}, 9.59\sqrt{2} \rangle.$$

Numerically,

$$\vec{v}\left(\frac{\pi}{4}\right) \approx \langle -13.576, 13.562 \rangle.$$

The speed is

$$\left| \vec{v}\left(\frac{\pi}{4}\right) \right| = \sqrt{(-19.2 \sin(\pi/4))^2 + (19.18 \cos(\pi/4))^2} = \sqrt{\frac{19.2^2 + 19.18^2}{2}} \approx 19.190.$$

At  $t = 0$ ,

$$\vec{v}(0) = \langle 0, 19.18 \rangle, \quad |\vec{v}(0)| = 19.18.$$

Therefore, under this parameterization, Uranus is moving *slightly faster* at  $t = \frac{\pi}{4}$  than at  $t = 0$ .

### B. Perihelion and Aphelion

The polar equation is

$$r(\theta) = \frac{a(1 - e^2)}{1 + e \cos \theta}.$$

Perihelion occurs at  $\theta = 0$ :

$$r_{\min} = \frac{a(1 - e^2)}{1 + e} = a(1 - e) = 19.2(1 - 0.047) = 18.2976.$$

So

$$\boxed{r_{\text{perihelion}} = 18.2976 \text{ AU.}}$$

Aphelion occurs at  $\theta = \pi$ :

$$r_{\max} = \frac{a(1 - e^2)}{1 - e} = a(1 + e) = 19.2(1 + 0.047) = 20.1024.$$

So

$$\boxed{r_{\text{aphelion}} = 20.1024 \text{ AU.}}$$

Now differentiate:

$$\frac{dr}{d\theta} = a(1 - e^2) \cdot \frac{e \sin \theta}{(1 + e \cos \theta)^2}.$$

Thus

$$\boxed{\frac{dr}{d\theta} = \frac{19.2(1 - 0.047^2)(0.047) \sin \theta}{(1 + 0.047 \cos \theta)^2}}.$$

To find where the distance from the Sun is changing most rapidly, maximize

$$\left| \frac{dr}{d\theta} \right|.$$

Setting the derivative equal to zero leads to

$$e \cos^2 \theta - \cos \theta - 2e = 0.$$

With  $e = 0.047$ , this gives

$$\cos \theta \approx -0.0936,$$

so

$$\theta \approx 1.665 \text{ radians} \approx 95.4^\circ.$$

By symmetry, the largest magnitude also occurs at

$$\theta \approx 2\pi - 1.665.$$

So the distance from the Sun is changing most rapidly at about

$$\boxed{\theta \approx 1.665 \text{ radians}}$$

(and symmetrically on the other side of the orbit).

## C. Area swept by Uranus

The polar area formula is

$$A = \frac{1}{2} \int r(\theta)^2 d\theta.$$

From  $\theta = 0$  to  $\theta = \frac{\pi}{2}$ ,

$$A_1 = \frac{1}{2} \int_0^{\pi/2} \left( \frac{a(1-e^2)}{1+e \cos \theta} \right)^2 d\theta.$$

From  $\theta = \frac{\pi}{2}$  to  $\theta = \pi$ ,

$$A_2 = \frac{1}{2} \int_{\pi/2}^{\pi} \left( \frac{a(1-e^2)}{1+e \cos \theta} \right)^2 d\theta.$$

So

$$A_1 = \frac{1}{2} \int_0^{\pi/2} \left( \frac{19.2(1-0.047^2)}{1+0.047 \cos \theta} \right)^2 d\theta$$

and

$$A_2 = \frac{1}{2} \int_{\pi/2}^{\pi} \left( \frac{19.2(1-0.047^2)}{1+0.047 \cos \theta} \right)^2 d\theta.$$

Since  $r(\theta)$  is generally larger on  $[\frac{\pi}{2}, \pi]$  than on  $[0, \frac{\pi}{2}]$ , the second area is larger:

$$A_2 > A_1.$$

This implies that Uranus is farther from the Sun near aphelion, so it moves more slowly there; near perihelion it moves faster.

## Part II: Detecting Neptune's Gravitational Tug

We are given

$$\vec{r}_1(t) = \langle 19.2 \cos t - 0.9, 19.18 \sin t \rangle$$

for Uranus, and

$$\vec{r}_2(t) = \langle 30.1 \cos(0.5t) - 0.27, 30.07 \sin(0.5t) \rangle$$

for Neptune.

### A. The Displacement Vector

The displacement vector from Uranus to Neptune is

$$\vec{d}(t) = \vec{r}_2(t) - \vec{r}_1(t).$$

So

$$\vec{d}(t) = \langle 30.1 \cos(0.5t) - 0.27 - (19.2 \cos t - 0.9), 30.07 \sin(0.5t) - 19.18 \sin t \rangle.$$

Simplifying,

$$\vec{d}(t) = \langle 30.1 \cos(0.5t) - 19.2 \cos t + 0.63, 30.07 \sin(0.5t) - 19.18 \sin t \rangle.$$

At  $t = 0$ ,

$$\vec{d}(0) = \langle 30.1 - 19.2 + 0.63, 0 \rangle = \langle 11.53, 0 \rangle.$$

Its magnitude is

$$|\vec{d}(0)| = 11.53.$$

So the unit vector is

$$\hat{u} = \frac{\vec{d}(0)}{|\vec{d}(0)|} = \langle 1, 0 \rangle.$$

Thus the direction of Neptune's pull at  $t = 0$  is

$$\langle 1, 0 \rangle,$$

meaning directly along the positive  $x$ -axis.

## B. Measuring the Wobble

The actual perturbed position is

$$\vec{r}_{\text{actual}}(t) = \langle 19.2 \cos t - 0.9 + 0.03 \sin(0.5t), 19.18 \sin t + 0.02 \cos(0.5t) \rangle.$$

So the deviation vector is

$$\vec{\varepsilon}(t) = \vec{r}_{\text{actual}}(t) - \vec{r}_1(t) = \langle 0.03 \sin(0.5t), 0.02 \cos(0.5t) \rangle.$$

Hence

$$|\vec{\varepsilon}(t)| = \sqrt{(0.03 \sin(0.5t))^2 + (0.02 \cos(0.5t))^2}.$$

Now evaluate:

$$|\vec{\varepsilon}(0)| = \sqrt{0^2 + (0.02)^2} = 0.02.$$

$$|\vec{\varepsilon}(\pi/2)| = \sqrt{(0.03 \sin(\pi/4))^2 + (0.02 \cos(\pi/4))^2} = \sqrt{0.00065} \approx 0.0255.$$

$$|\vec{\varepsilon}(\pi)| = \sqrt{(0.03 \sin(\pi/2))^2 + (0.02 \cos(\pi/2))^2} = 0.03.$$

$$|\vec{\varepsilon}(3\pi/2)| = \sqrt{(0.03 \sin(3\pi/4))^2 + (0.02 \cos(3\pi/4))^2} = \sqrt{0.00065} \approx 0.0255.$$

So:

$$|\vec{\varepsilon}(0)| = 0.020, \quad |\vec{\varepsilon}(\pi/2)| \approx 0.0255, \quad |\vec{\varepsilon}(\pi)| = 0.030, \quad |\vec{\varepsilon}(3\pi/2)| \approx 0.0255$$

The deviation is largest at

$$t = \pi.$$

This makes physical sense because the perturbation has built up to its maximum at that point in the oscillation.

### C. Rate of change of the deviation

Differentiate:

$$\vec{\varepsilon}'(t) = \left\langle 0.03 \cdot \frac{1}{2} \cos(0.5t), 0.02 \cdot \left(-\frac{1}{2} \sin(0.5t)\right) \right\rangle.$$

So

$$\boxed{\vec{\varepsilon}'(t) = \langle 0.015 \cos(0.5t), -0.01 \sin(0.5t) \rangle.}$$

Its magnitude is

$$|\vec{\varepsilon}'(t)| = \sqrt{(0.015 \cos(0.5t))^2 + (-0.01 \sin(0.5t))^2}.$$

This is largest when  $\cos(0.5t) = \pm 1$  and  $\sin(0.5t) = 0$ , so

$$\boxed{t = 0, 2\pi, 4\pi, \dots}$$

Thus the deviation changes most rapidly at the start of each full wobble cycle.

An astronomer would notice the wobble accelerating and decelerating because the perturbation is oscillatory: sometimes the deviation changes quickly, and sometimes it changes more slowly. Tracking both the size and direction of this wobble would help reveal the direction of the unseen planet's gravitational pull.

## Prediction: Sample Telescope Recommendation

### Sample response:

The orbital analysis suggests that Uranus is not following a perfectly predicted path. In Part I, the parametric and polar models show how Uranus should move if only the Sun's gravity were acting. In Part II, however, the deviation vector

$$\vec{\varepsilon}(t) = \langle 0.03 \sin(0.5t), 0.02 \cos(0.5t) \rangle$$

shows that Uranus's actual position differs from its expected position by a small but measurable amount. The size of this deviation reaches a maximum of 0.03 AU at  $t = \pi$ , which is strong evidence that another planet is exerting a gravitational influence.

The displacement vector from Uranus to Neptune at  $t = 0$  gives the unit vector

$$\langle 1, 0 \rangle,$$

so the unseen planet appears to lie in the positive  $x$ -direction from Uranus at that moment. For that reason, a telescope should be pointed in the direction of Uranus's observed pull, especially along the positive  $x$ -axis relative to Uranus's position. This is the most likely direction in which to search for the new planet.