Solutions: Cauchy's Calculus

Part I: Limits

1. Cauchy's definition of limits

a) In modern language, Cauchy means that as a variable takes values closer and closer to some fixed number, the difference between the variable and that number can be made as small as we want. To "indefinitely approach" a fixed value means the values continue getting nearer and nearer to it. "Differing from it by as little as we wish" means that eventually the difference can be made smaller than any positive amount we choose.

b) The sequence is

$$a_n = \frac{2n+1}{n+1}.$$

Compute:

$$a_1 = \frac{2(1)+1}{1+1} = \frac{3}{2}, \qquad a_{10} = \frac{21}{11}, \qquad a_{100} = \frac{201}{101}, \qquad a_{1000} = \frac{2001}{1001}.$$

Numerically,

$$\frac{3}{2} = 1.5, \qquad \frac{21}{11} \approx 1.909, \qquad \frac{201}{101} \approx 1.990, \qquad \frac{2001}{1001} \approx 1.999.$$

The sequence appears to approach 2.

c)

$$\lim_{n\to\infty} \frac{2n+1}{n+1}.$$

Divide numerator and denominator by n:

$$\frac{2n+1}{n+1} = \frac{2+\frac{1}{n}}{1+\frac{1}{n}}.$$

As $n \to \infty$, $\frac{1}{n} \to 0$, so

$$\lim_{n\to\infty} \frac{2n+1}{n+1} = \frac{2+0}{1+0} = 2.$$

d) The distance from a_n to the limit 2 is

$$\left|\frac{2n+1}{n+1} - 2\right| = \left|\frac{2n+1-2n-2}{n+1}\right| = \frac{1}{n+1}.$$

So

$$|a_{100} - 2| = \frac{1}{101}, \qquad |a_{1000} - 2| = \frac{1}{1001}.$$

Since these distances get smaller as n increases, this illustrates Cauchy's phrase "differing from it by as little as we wish": by taking n large enough, the sequence can be made arbitrarily close to 2.

2. Infinitesimals

a) In modern language, Cauchy is describing a sequence whose limit is 0. Saying the absolute values become less than any given quantity means that for every positive number ε, eventually the terms satisfy $|a_n| < \varepsilon$. That is exactly what it means for a sequence to converge to 0.

b) (i) $a_n = \dfrac{1}{n^2}$

First four terms:

$$a_1 = 1, \quad a_2 = \frac{1}{4}, \quad a_3 = \frac{1}{9}, \quad a_4 = \frac{1}{16}.$$

Since $\frac{1}{n^2} \to 0$, this is an infinitesimal.

(ii) $b_n = \dfrac{(-1)^n}{n}$

First four terms:

$$b_1 = -1, \quad b_2 = \frac{1}{2}, \quad b_3 = -\frac{1}{3}, \quad b_4 = \frac{1}{4}.$$

Since

$$\left|\frac{(-1)^n}{n}\right| = \frac{1}{n} \to 0,$$

this is an infinitesimal.

(iii) $c_n = (-1)^n$

First four terms:

$$c_1 = -1, \quad c_2 = 1, \quad c_3 = -1, \quad c_4 = 1.$$

This does not approach 0, so it is not an infinitesimal.

c) The alternating sign does not matter for b_n because Cauchy's definition uses absolute values. Since

$$|b_n| = \frac{1}{n} \to 0,$$

the terms become arbitrarily small in magnitude. By contrast,

$$|c_n| = 1$$

for every n, so c_n never becomes smaller than an arbitrary positive number. Therefore c_n fails Cauchy's definition.

3. Bounding arguments

a) If

$$-x^2 \leq f(x) \leq x^2$$

for all x near 0, then since

$$\lim_{x\to 0}(-x^2) = 0 \qquad \text{and} \qquad \lim_{x\to 0} x^2 = 0,$$

it follows that

$$\lim_{x\to 0} f(x) = 0.$$

The theorem is the **Squeeze Theorem**.

b) Evaluate

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right).$$

Since

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1,$$

for all $x \neq 0$, multiplying by $x^2 \geq 0$ gives

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2.$$

Because

$$\lim_{x\to 0}(-x^2) = 0 \qquad \text{and} \qquad \lim_{x\to 0} x^2 = 0,$$

the Squeeze Theorem implies

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

Part II: Continuity

4. Cauchy's definition of continuity

a) Cauchy means that if the input changes by a very small amount, then the output should also change by a very small amount. The "infinitely small increment in the variable" is a tiny change in x, and the corresponding "infinitely small increment" in the function is the tiny change in $f(x)$. So continuity means small changes in input do not produce sudden jumps in output.

b) The three modern conditions are:

  i) $f(a)$ is defined.

  ii) $\lim_{x\to a} f(x)$ exists.

iii) $\lim_{x \to a} f(x) = f(a)$.

These match Cauchy's idea because the function must have a value at the point, nearby values must approach a single number, and that number must agree with the actual function value so that no jump or break occurs.

c) Let

$$f(x) = x^2 + 3x.$$

Check continuity at $x = 1$:

i)

$$f(1) = 1^2 + 3(1) = 4,$$

so $f(1)$ is defined.

ii) Since f is a polynomial,

$$\lim_{x \to 1}(x^2 + 3x) = 1^2 + 3(1) = 4.$$

iii)

$$\lim_{x \to 1} f(x) = 4 = f(1).$$

Therefore $f(x) = x^2 + 3x$ is continuous at $x = 1$.

5. Jump discontinuity

a) A function has a **jump discontinuity** at $x = a$ if both one-sided limits exist but are not equal:

$$\lim_{x \to a^-} f(x) \quad \text{and} \quad \lim_{x \to a^+} f(x)$$

both exist, but

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x).$$

b) The piecewise function is

$$f(x) = x + 2 \quad \text{when } x < 2, \qquad f(x) = x^2 - 1 \quad \text{when } x \geq 2.$$

Compute the one-sided limits:

$$\lim_{x \to 2^-} f(x) = 2 + 2 = 4,$$

$$\lim_{x \to 2^+} f(x) = 2^2 - 1 = 3.$$

Since

$$\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x),$$

the two-sided limit at $x = 2$ does not exist. Therefore f is **not continuous** at $x = 2$.

Because both one-sided limits exist but are different, the discontinuity is a **jump discontinuity**.

c) In Cauchy's terms, this function fails to be continuous at $x = 2$ because a very small change in the input near 2 can produce a sudden non-small change in the output. As x approaches 2 from the left, $f(x)$ approaches 4, but as x approaches from the right, $f(x)$ approaches 3. So when the input crosses 2, the output jumps instead of changing by an infinitesimal amount.

6. Unbounded behavior

a) Consider

$$f(x) = \frac{1}{x}.$$

At $x = 0.01$:

i)

$$f(0.01) = \frac{1}{0.01} = 100,$$

so $f(0.01)$ is defined.

ii) Since $\frac{1}{x}$ is continuous for all $x \neq 0$,

$$\lim_{x \to 0.01} \frac{1}{x} = 100.$$

iii)

$$\lim_{x \to 0.01} \frac{1}{x} = 100 = f(0.01).$$

So $f(x) = \dfrac{1}{x}$ is continuous at $x = 0.01$.

b) As $x \to 0^+$,

$$\frac{1}{x} \to +\infty.$$

As $x \to 0^-$,

$$\frac{1}{x} \to -\infty.$$

So the function becomes unbounded near $x = 0$. This is an **infinite discontinuity**.

7. Intermediate Value Theorem

a) Since f is continuous on $[0, 5]$, and

$$f(0) = -2, \qquad f(5) = 8,$$

the function must take every value between -2 and 8 somewhere on the interval. Since 3 lies between -2 and 8, we can guarantee that there is some $x \in [0, 5]$ such that

$$f(x) = 3.$$

However, 10 does not lie between -2 and 8, so we cannot guarantee that $f(x) = 10$ anywhere on $[0, 5]$.

The theorem is the **Intermediate Value Theorem**.

b) Let

$$f(x) = x^3 - 4x + 1.$$

We want to show that f has a root in $[1, 2]$.

First, f is a polynomial, so it is continuous on $[1, 2]$.

Next compute:

$$f(1) = 1^3 - 4(1) + 1 = -2,$$
$$f(2) = 2^3 - 4(2) + 1 = 1.$$

Since

$$f(1) < 0 \qquad \text{and} \qquad f(2) > 0,$$

f changes sign on $[1, 2]$.

By the Intermediate Value Theorem, because f is continuous on $[1, 2]$ and 0 lies between $f(1)$ and $f(2)$, there must be some $c \in (1, 2)$ such that

$$f(c) = 0.$$

Therefore f has at least one root in $[1, 2]$.