Solutions: Apollo 13 Re-Entry

Part I: Verifying the Gravity Model

We are given

$$\frac{dv}{dx} = \frac{-gR^2}{v(R+x)^2},$$

where

$$g = 32 \text{ ft/s}^2, \qquad R = 20{,}900{,}000 \text{ ft},$$

and the initial condition

$$v(400{,}000) = 36{,}100 \text{ ft/s}.$$

A. Derive the velocity model

Start by separating variables:

$$v\,dv = -\frac{gR^2}{(R+x)^2}\,dx.$$

Integrate both sides:

$$\int v\,dv = \int -\frac{gR^2}{(R+x)^2}\,dx.$$

This gives

$$\frac{1}{2}v^2 = \frac{gR^2}{R+x} + C.$$

Use the initial condition $v(400{,}000) = 36{,}100$:

$$\frac{1}{2}(36{,}100)^2 = \frac{32(20{,}900{,}000)^2}{20{,}900{,}000 + 400{,}000} + C.$$

So the model can be written as

$$\frac{1}{2}v^2 = \frac{32R^2}{R+x} + \left(\frac{1}{2}(36{,}100)^2 - \frac{32R^2}{R+400{,}000}\right).$$

Solving for v,

$$v(x) = \sqrt{(36{,}100)^2 + 2\cdot 32\cdot R^2\left(\frac{1}{R+x} - \frac{1}{R+400{,}000}\right)}.$$

Therefore,

$$\boxed{v(x) = \sqrt{(36{,}100)^2 + 64R^2\left(\frac{1}{R+x} - \frac{1}{R+400{,}000}\right)}}.$$

B. Predicted re-entry speed at $x = 300{,}000$ ft

Substitute $x = 300{,}000$:

$$v(300{,}000) = \sqrt{(36{,}100)^2 + 64(20{,}900{,}000)^2 \left(\frac{1}{20{,}900{,}000 + 300{,}000} - \frac{1}{20{,}900{,}000 + 400{,}000} \right)}.$$

Evaluating,

$$v(300{,}000) \approx 36{,}185.646 \text{ ft/s}.$$

So,

$$\boxed{v(300{,}000) \approx 36{,}185.646 \text{ ft/s}}.$$

The safe corridor is between 36,100 ft/s and 36,400 ft/s, so this value *is within the safe range*. Therefore, the predicted speed is safe.

C. Quantitative Error Propagation

1. Compute $\frac{dx}{dv}$ at $x = 300{,}000$

From

$$\frac{dv}{dx} = \frac{-gR^2}{v(R+x)^2},$$

invert to get

$$\frac{dx}{dv} = \frac{1}{dv/dx} = -\frac{v(R+x)^2}{gR^2}.$$

At $x = 300{,}000$ and $v \approx 36{,}185.646$,

$$\frac{dx}{dv} = -\frac{(36{,}185.646)(20{,}900{,}000 + 300{,}000)^2}{32(20{,}900{,}000)^2} \approx -1163.498.$$

Thus,

$$\boxed{\frac{dx}{dv} \approx -1163.498 \text{ s}}.$$

If $dv = \pm 2$ ft/s, then

$$dx \approx \frac{dx}{dv}\, dv \approx (-1163.498)(\pm 2) \approx \mp 2326.995 \text{ ft}.$$

So the altitude uncertainty is about

$$\boxed{|dx| \approx 2327 \text{ ft}}.$$

2. Interpretation

A velocity error of only 2 ft/s could shift the predicted entry altitude by about 2327 ft vertically. That means the atmospheric entry point could be off by nearly half a mile, which shows that even a tiny speed error can translate into a noticeable positional error.

Part II: The Parachute Descent

We are given

$$\frac{dv}{dt} = 32 - 0.15v, \qquad v(0) = 400.$$

A. Euler's Method with $\Delta t = 0.5$

Euler's update rule is

$$v_{\text{new}} = v_{\text{old}} + (32 - 0.15v_{\text{old}})\,(0.5).$$

t (s)	v (ft/s)	Computation
0.0	400.000	given
0.5	386.000	$400 + (32 - 0.15(400))(0.5)$
1.0	373.050	$386 + (32 - 0.15(386))(0.5)$
1.5	361.071	$373.05 + (32 - 0.15(373.05))(0.5)$
2.0	349.991	$361.07125 + (32 - 0.15(361.07125))(0.5)$

So the Euler estimate at $t = 2$ is

$$\boxed{v(2) \approx 349.991 \text{ ft/s}}.$$

B. Accuracy check

The exact value is given as

$$v(2) = 351.625 \text{ ft/s}.$$

So the absolute error is

$$|351.625 - 349.991| = 1.634.$$

Thus,

$$\boxed{\text{Absolute error} \approx 1.634 \text{ ft/s}}.$$

This error would likely have been acceptable because it is very small compared to the overall speed of the spacecraft. The error is also relatively small because:

- the time interval is short (0 to 2 seconds),
- the step size is fairly small (0.5 seconds),
- the differential equation is smooth and linear, so Euler's Method performs reasonably well.

C. Computational Constraints

NASA engineers would choose Euler's Method because it is simple, fast, and uses very little memory. Each step only requires the current velocity and a basic arithmetic update, so the AGC would not need to store a large table of values or carry out more advanced calculations in real time. For a computer with extremely limited memory and processing power, Euler's Method was a practical compromise between accuracy and efficiency.

Flight Readiness Certification

Sample Memo

To: NASA Flight Control

Based on the gravity model, the spacecraft's predicted velocity at the entry altitude of 300,000 ft is approximately 36,185.646 ft/s. This lies safely within the acceptable re-entry corridor of 36,100 ft/s to 36,400 ft/s, so the trajectory meets the required speed condition for safe atmospheric entry.

The calculations also suggest that the prediction is reasonably reliable. In Part I-C, a velocity uncertainty of only ± 2 ft/s produced an altitude uncertainty of about ± 2327 ft, showing that small speed errors do affect the predicted entry point, but not enough here to push the spacecraft outside the safe corridor. In Part II-B, Euler's Method gave an approximation within about 1.634 ft/s of the exact value, indicating that the AGC's numerical methods were accurate enough for this purpose. Based on these results, I certify that this Apollo 13 trajectory is **flight-ready**.