

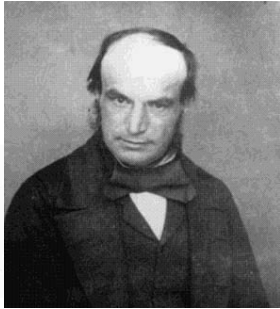
## Project: Finding Neptune

**Unit: Parametric Equations, Polar Coordinates, and Vector-Valued Functions (AP Calc BC Unit 9)**

**Topics: Derivatives of parametric and vector-valued functions, Velocity, speed, and acceleration along a curve**

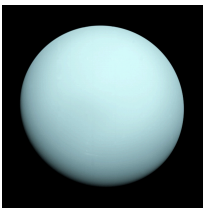
In this project, students investigate orbital anomalies that led to the discovery of Neptune by analyzing parametric and polar models of planetary orbits, computing velocity and displacement vectors, and tracking gravitational perturbations.

### Introduction:



It's 1845, and you're a mathematician who has noticed something wrong with Uranus. The planet's observed orbit doesn't match the orbit predicted by Newton's laws. You have decided to determine whether an undiscovered eighth planet might explain the discrepancy, and, if so, to try to find it.

### Part I: Modeling and Comparing Orbits



To predict where Uranus should be, you can model its orbit as an ellipse. The following parametric equations give Uranus's position in the orbital plane (in AU), where  $t$  represents the orbital angle in radians:

$$x(t) = 19.2\cos(t) - 0.9, y(t) = 19.18\sin(t), 0 \leq t \leq 2\pi$$

The same orbit can be written in polar form, where  $r(\theta)$  gives Uranus's distance from the Sun at angle  $\theta$ , with the sun at the origin:

$$r(\theta) = \frac{a(1-e^2)}{1+e\cos(\theta)}$$

where  $a = 19.2$  AU and  $e = 0.047$

### A. Velocity along the orbit

Find  $dx/dt$  and  $dy/dt$  from the parametric model. Then, compute the velocity vector and speed at  $t = \frac{\pi}{4}$ . Is Uranus moving faster or slower here than at  $t = 0$ ? Explain why this makes sense for an elliptical orbit.

### B. Perihelion and Aphelion

Using the polar equation, compute the perihelion (the closest distance to the Sun) and aphelion (the farthest distance from the Sun). Then find  $dr/d\theta$  and determine at what angle the distance from the Sun is changing most rapidly.

### C. Area swept by Uranus

Kepler's Second Law says a planet sweeps out equal areas in equal times. Set up (but do not evaluate) the polar area integrals for the region swept from  $\theta = 0$  to  $\theta = \frac{\pi}{2}$  and from  $\theta = \frac{\pi}{2}$  to  $\theta = \pi$ . Without computing them, explain which area is larger and what this implies about how fast Uranus moves at different points in its orbit.

## Part II: Detecting Neptune's Gravitational Tug

Neptune's gravity is what pulls Uranus off its predicted path. To model this, suppose at time  $t$ , Uranus is at:

$$r_1(t) = \langle 19.2\cos(t) - 0.9, 19.18\sin(t) \rangle$$

And Neptune (which orbits more slowly) is at:

$$r_2(t) = \langle 30.1\cos(0.5t) - 0.27, 30.07\sin(0.5t) \rangle$$

### A. The Displacement Vector

Write the displacement vector  $d(t) = r_2(t) - r_1(t)$  pointing from Uranus toward Neptune.

At  $t = 0$ , find the unit vector in this direction. This is the direction of Neptune's gravitational pull on Uranus.

### B. Measuring the Wobble

Suppose Uranus's actual (perturbed) position is:

$$r_{actual}(t) = \langle 19.2\cos(t) - 0.9 + 0.03\sin(0.5t), 19.18\sin(t) + 0.02\cos(0.5t) \rangle$$

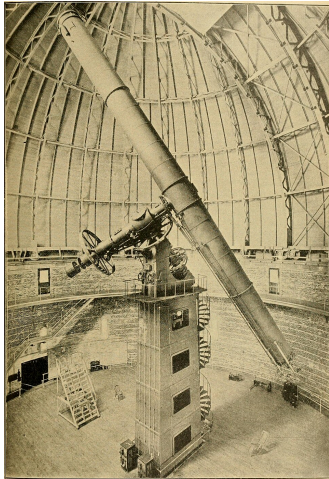
Write the deviation vector  $\epsilon(t) = r_{actual}(t) - r_1(t)$ . Compute  $|\epsilon(t)|$  at  $t = 0, \frac{\pi}{2}, \pi,$  and  $\frac{3\pi}{2}$ .

When is the deviation largest, and why does this make physical sense given Neptune's position?

### C. Rate of change of the deviation

Find  $\epsilon'(t)$ . At what time is the deviation changing most rapidly? Explain why an astronomer tracking Uranus over time would notice the wobble accelerating and decelerating, and how this pattern reveals the direction of the unseen planet.

#### Prediction:



Now that you have completed your analysis, write a short 1-2 paragraph report recommending where a telescope should be pointed to find the new planet.

Make sure to include:

1. Evidence from your orbital analysis (Part I) and deviation analysis (Part II) that supports the existence of an unseen planet.
2. Your estimate of the direction of Neptune's pull on Uranus, citing your unit vector from Part II-A.